Final - Computer Science 2 (2020-21)
Time: 3 hours.
Attempt all questions, giving proper explanations.
You may quote any result proved in class without proof.

1. How is the number 22.5 stored as a floating point number in the computer ? Give the sign, mantissa and exponent. **[4 marks]**

2. (a) How many floating point numbers are there in $[0, 1]$? **[2 marks]**
   (b) How many floating point numbers are there in [1, `.Machine$double.xmax`]? **[2 marks]**
   (c) Are the number of floating point numbers in $[0, 1]$ equal to the number of floating point numbers in $2 + [0, 1] = [2, 3]$? Explain. **[2 marks]**

3. Consider the solution to $x = g(x)$ in $[0, 1]$ where $g(x) = \frac{1}{4}(1 - x)^4$. Consider the iterations $x_{k+1} = g(x_k)$. Starting from $x_0 = \frac{1}{2}$ roughly how many iterations are necessary before we are within 10^{-6} of the solution? **[4 marks]**

4. Consider the equation $x^2 - 3 = 0$.
   (a) Show, starting from $x_0 = 1$, that Newton's method converges to $\sqrt{3}$. **[4 marks]**
   (b) For what values of x_0 does Newton's method converge to $\sqrt{3}$? It is enough to give a non-rigorous explanation. **[2 marks]**

5. Consider the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ given below.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix}$$

   (a) Find the QR decomposition of $\mathbf{A}$. **[4 marks]**
   (b) Find the least squares solution of the system $\mathbf{Ax} = \mathbf{b}$. **[2 marks]**

6. Consider an infinitely differentiable function $f : [0, 1] \to \mathbf{R}$.
   (a) Write down the Newton-Cotes formula for $\int_0^1 f(x)dx$ with 4 equally spaced points $0 = x_0 < x_1 < x_2 < x_3 = 1$. **[4 marks]**
   (b) What is the error in approximating the integral by the approximation? **[2 marks]**

7. For this problem you may use `R` for some of the computations.
   (a) Find a set of points $x_0, x_1, x_2 \in [0, 1]$ such that for *any* $f \in \mathcal{P}_5$, the set of polynomials of degree at most 5, we have

$$\int_0^1 f(x)dx = \int_0^1 p_2(x)dx$$

   where $p_2(x)$ is the Lagrange polynomial passing through the points $(x_i, f(x_i))$, $i = 0, 1, 2$. **[5 marks]**
   Note: If you require to find the roots of a polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, type `polyroot(z=c(`$a_0, a_1, a_2, \cdots, a_n$`))` in `R`.
   (b) For any $f \in \mathcal{P}_5$ write down $\int_0^1 f(x)dx$ as a linear combination of $f(x_i)$, $i = 0, 1, 2$, where x_i are computed above. **[3 marks]**
   (c) Explain why this quadrature rule is better than the Simpson quatrature rule. **[1 mark]**

8. Consider the solution $\big(x(t), y(t)\big)$, $0 \le t \le \frac{3}{4}$, to the *system* of differential equations:

$$\frac{dx}{dt} = (x+2)^2, \quad x(0) = -1,$$
$$\frac{dy}{dt} = x, \qquad\qquad y(0) = 0.$$

For $h > 0$, we use the Euler approximation :

$$x_{i+1} = x_i + h(x_i + 2)^2,$$
$$y_{i+1} = y_i + h x_i.$$

with $x_0 = -1$, $y_0 = 0$ and (x_i, y_i) the approximation at the point $t_i = ih$. Show that

$$y_{[\frac{1}{2h}]+1} - y_{[\frac{1}{2h}]} = O(h^2).$$

(Above $[a]$ denotes the integer part of a.) **[4 marks]**